MEASURE OF FULL DIMENSION FOR SOME NONCONFORMAL REPELLERS

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ABSTRACT

Given (X,T) and (Y,S) mixing subshifts of finite type such that (Y,S) is a factor of (X,T) with factor map $\pi\colon X\to Y$, and positive Hölder continuous functions $\varphi\colon X\to\mathbb{R}$ and $\psi\colon Y\to\mathbb{R}$, we prove that the maximum of

$$\frac{h_{\mu\circ\pi^{-1}}(S)}{\int\psi\circ\pi\,d\mu}+\frac{h_\mu(T)-h_{\mu\circ\pi^{-1}}(S)}{\int\varphi\,d\mu}$$

over all T-invariant Borel probability measures μ on X is attained on the subset of ergodic measures. Here $h_\mu(T)$ stands for the metric entropy of T with respect to μ. As an application, we prove the existence of an ergodic invariant measure with full dimension for a class of transformations treated in [GP1], and also for the transformations treated in [L1], where the author considers nonlinear skew-product perturbations of *general Sierpinski carpets*. In order to do so we establish a variational principle for the topological pressure of certain noncompact sets.

1. Introduction

The problem we are interested in is the computation of Hausdorff dimension of invariant sets for dynamical systems, and moreover the existence of an ergodic invariant measure on the invariant set with full dimension. We are particularly interested in the case of a map f and f-invariant sets Λ such that $f|\Lambda$ is *expanding*. Even in this case, it is still a widely open problem only solved in great generality when $f|\Lambda$ is expanding and conformal, due to the thermodynamic formalism introduced by Sinai, Ruelle and Bowen (see [B2], [B3], [R2]): there is a formula in terms of the *pressure function* for the Hausdorff dimension, and there is an ergodic invariant measure of full dimension which is a *Gibbs state*.

In the nonconformal setting the computation of Hausdorff dimension is more delicate due to the existence of several rates of expansion, and no such general formula exists. In [L1] we consider a class of skew-product expanding maps of the 2-torus of the form $f(x,y)=(a(x,y),b(y))$ satisfying $\inf|Da|>\sup|Db|$, and consider invariant sets Λ which possess a simple Markov structure, proving the *variational principle for Hausdorff dimension*, i.e the existence of ergodic invariant measures on Λ with dimension arbitrarily close to the dimension of Λ. Now because of the nonlinearity of the maps the problem of finding a maximizing ergodic measure turns into another nontrivial problem. This is because although we have a sequence of ergodic invariant measures whose dimension converges to the dimension of the invariant set, the map $\mu\mapsto\dim_{\mathrm{H}}\mu$ is not in general upper-semicontinuous. In [L2] we exhibit an open class of repellers, that possess a *dominating spliting* that is *not too strong*, for which there is an ergodic measure of full dimension. This is done by showing that the methods of Heuter and Lalley [HL], who prove the validity of Falconer's formula (see [F2]) for an open class of self-affine transformations, extend to nonlinear transformations. In this work we prove the existence of an ergodic

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invariant measure with full dimension for the class of transformations treated in [L1], thus giving new insight for solving this kind of problem for nonlinear transformations. The novelty is in reducing the problem to the the variational principle for the topological pressure for certain noncompact sets, which we establish.

Now we give an overview of related results, culminating in [L1] and this work. The simplest examples of invariant sets for nonconformal transformations are the *general Sierpinski carpets*. These are invariant sets for transformations of the 2-torus of the form $f(x, y) = (lx, my)$, where $l > m > 1$ are integers, and are modeled by a full shift. Bedford [Be] and McMullen [Mu] proved independently a formula for the Hausdorff dimension of these sets which coincides with the dimension of a Bernoulli measure. Later, Gatzouras and Lalley [GL] generalized these results for certain self-affine sets. Until then only sets modeled by a full shift were treated when Kenyon and Peres [KP] considered expanding linear endomorphisms of the d-torus which are a direct sum of conformal endomorphisms, and proved that any invariant set supports an ergodic invariant measure of full dimension, thus solving the problem for these kind of transformations. Following [GL], Gatzouras and Peres [GP1] considered nonlinear maps of the form $f(x, y) = (f_1(x), f_2(y))$, where f_1 and f_2 are conformal and expanding maps satisfying $\inf |Df_1| \geq \sup |Df_2|$, and showed that, for a large class of invariant sets Λ, there exist ergodic invariant measures with dimension arbitrarily close to the dimension of Λ. The methods used in this work also provide an invariant ergodic measure with full dimension for a class of tranformations treated in [GP1].

The notion of topological entropy for noncompact sets was introduced by Bowen [B1]. In [PP] Pesin and Pitskel introduced the notion of topological pressure for noncompact sets and proved the variational principle for the topological pressure for certain noncompact sets. The noncompact sets we are interested in are the ones treated in [BS] and [TV] where they prove the variational principle for the topological entropy for these sets.

Let us now state the main result. Given a continuous map $T: X \to X$ of a compact metric space X, we use the following notation: $\mathcal{M}(T)$ is the set of all T-invariant Borel probability measures on X; $\mathcal{M}_e(T) \subset \mathcal{M}(T)$ is the subset of ergodic measures; $h_\mu(T)$ is the metric entropy of T with respect to $\mu \in \mathcal{M}(T)$.

Theorem A. *Let (X, T) and (Y, S) be mixing subshifts of finite type, and $\pi: X \to Y$ be a continuous and surjective mapping such that $\pi \circ T = S \circ \pi$ (S is a factor of T). Let $\varphi: X \to \mathbb{R}$ and $\psi: Y \to \mathbb{R}$ be positive Hölder continuous functions. Then the maximum of*

$$\frac{h_{\mu\circ\pi^{-1}}(S)}{\int \psi \circ \pi \, d\mu} + \frac{h_\mu(T) - h_{\mu\circ\pi^{-1}}(S)}{\int \varphi \, d\mu} \tag{1}$$

over all $\mu \in \mathcal{M}(T)$ is attained on the set $\mathcal{M}_e(T)$.

Remark 1. Theorem A answers positively to Problem 2 raised in [GP2] for mixing subshifts of finite type and Hölder continuous potentials. So, it applies to obtain an invariant ergodic measure of full dimension for a class of transformations treated in [GP1].

Now we define *general Sierpinski carpet.* Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus and $f_0: \mathbb{T}^2 \to \mathbb{T}^2$ be given by $f_0(x, y) = (lx, my)$ where $l > m > 1$ are integers. The grid of lines $[0, 1] \times \{i/m\}$, $i = 0, ..., m-1$, and $\{j/l\} \times [0, 1]$, $j = 0, ..., l-1$, form a set of rectangles each of which is mapped by f_0 onto the entire torus (these rectangles are the domains of invertibility of f_0). Now choose some of these rectangles and consider the fractal set Λ_0 consisting of those points that always remain in these chosen rectangles when iterating f_0. Geometrically, Λ_0 is the limit (in the Hausdorff metric) of n-*approximations*: the 1-approximation consists of the chosen rectangles,

the 2-approximation consists in dividing each rectangle of the 1-aproximation into $l \times m$ subrectangles and selecting those with the same pattern as in the begining, and so on. We say that (f_0, Λ_0) is a *general Sierpinski carpet*.

Let (f_0, Λ_0) be a general Sierpinski carpet. There exists $\varepsilon > 0$ such that if f is $\varepsilon - \mathrm{C}^1$ close to f_0 then there is a unique homeomorphism $h\colon \mathbb{T}^2 \to \mathbb{T}^2$ close to the identity which conjugates f and f_0, i.e $f \circ h = h \circ f_0$ (see [S]).

Definition 1. The f-invariant set $\Lambda = h(\Lambda_0)$ is called the f-*continuation* of Λ_0.

Definition 2. Let $\mathcal{S}$ be the class of C^2 maps $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ of the form

$$f(x, y) = (a(x, y), b(y)).$$

Notation: $\dim_{\mathrm{H}} \Lambda$ and $\dim_{\mathrm{H}} \mu$ stand for the Hausdorff dimension of a set Λ and a measure μ, respectively.

Then the following is proved in [L1].

Theorem 1. *Let (f_0, Λ_0) be a general Sierpinski carpet. There exists $\varepsilon > 0$ such that if $f \in \mathcal{S}$ is $\varepsilon - \mathrm{C}^2$ close to f_0, and Λ is the f-continuation of Λ_0, then*

$$\dim_{\mathrm{H}} \Lambda = \sup\{\dim_{\mathrm{H}} \mu : \mu(\Lambda) = 1,\ \mu \textit{ is } f\textit{-invariant and ergodic}\}.$$

Here we improve this result.

Theorem B. *With the same hypothesis of Theorem 1, there exists an ergodic invariant measure μ on Λ such that*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} \mu.$$

Morever, μ is a Gibbs state for a relativized variational principle.

To prove these results we use the variational principle for the topological pressure for certain noncompact sets as we shall describe now. Given a continuous map $T\colon X \to X$ of a compact metric space,denote by $P(\psi, K)$ the topological pressure associated to a continuous function $\psi\colon X \to \mathbb{R}$ and a T-invariant set K (not necessarly compact), as defined in [PP]. Let

$$I_\psi = \left(\inf_{\mu \in \mathcal{M}(T)} \int \psi \, d\mu, \sup_{\mu \in \mathcal{M}(T)} \int \psi \, d\mu \right).$$

Theorem C. *Let (X, T) be a mixing subshift of finite type, and $\varphi, \psi\colon X \to \mathbb{R}$ Hölder continuous functions. For $\alpha \in \mathbb{R}$ let*

$$K_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(T^i(x)) = \alpha \right\}.$$

If $0 \notin \partial I_\psi$ and $\alpha \in I_\psi$ then

$$P(\varphi, K_\alpha) = \sup\left\{ h_\mu(T) + \int \varphi \, d\mu : \mu \in \mathcal{M}(T) \textit{ and } \int \psi \, d\mu = \alpha \right\}.$$

Morever, the supremum is attained at a unique measure μ_β which is the Gibbs state with respect to the potential $\varphi + \beta\psi$, for a unique $\beta \in \mathbb{R}$.

For definitions and basic results about dimension we refer the reader to the books [F1] and [P].

2. Proof of Theorem C

We start by proving the 'moreover' part. We have, for all $\beta \in \mathbb{R}$,

$$\begin{aligned}
&\sup\left\{h_\mu(T) + \int \varphi\, d\mu : \mu \in \mathcal{M}(T) \text{ and } \int \psi\, d\mu = \alpha\right\} \\
&= \sup\left\{h_\mu(T) + \int \varphi\, d\mu + \beta\int \psi\, d\mu : \mu \in \mathcal{M}(T) \text{ and } \int \psi\, d\mu = \alpha\right\} - \beta\alpha \\
&\leq \sup\left\{h_\mu(T) + \int \varphi\, d\mu + \beta\int \psi\, d\mu : \mu \in \mathcal{M}(T)\right\} - \beta\alpha.
\end{aligned}$$

Now it is well known (see [B2]) that the last supremum is uniquely attained at the *Gibbs state* μ_β associated to the potential $\varphi + \beta\psi$ (for the classical variational principle). So we must find a unique β such that $\int \psi\, d\mu_\beta = \alpha$.

We use the abbreviation $P(\cdot) = P(\cdot, X)$. It is proved in [R1] that $P(\cdot)$ is a real analytic function on the space of Hölder continuous functions and that

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} P(h_1 + \varepsilon h_2) = \int h_2\, d\mu_{h_1}, \\
&\frac{\partial^2 P(h + \varepsilon_1 h_1 + \varepsilon_2 h_2)}{\partial\varepsilon_1 \partial\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} = Q_h(h_1, h_2),
\end{aligned}$$

where Q_h is the bilinear form defined by

$$Q_h(h_1, h_2) = \sum_{n=0}^{\infty} \left(\int h_1 (h_2 \circ f^n)\, d\mu_h - \int h_1\, d\mu_h \int h_2\, d\mu_h\right), \tag{2}$$

and μ_h is the Gibbs measure for the potential h. Moreover, $Q_h(h_1, h_1) \geq 0$ and $Q_h(h_1, h_1) = 0$ if and only if h_1 is *cohomologous* to a constant function. From this we get that

$$\frac{d}{d\beta}\int \psi\, d\mu_\beta = \frac{d^2}{d\beta^2} P(\varphi + \beta\psi) = Q_{\varphi+\beta\psi}(\psi, \psi) > 0 \tag{3}$$

(the hypothesis $\alpha \in I_\psi$ prevents ψ being cohomologous to a constant). So we must see that

$$\lim_{\beta\to\infty} \int \psi\, d\mu_\beta = \sup_{\mu \in \mathcal{M}(T)} \int \psi\, d\mu, \tag{4}$$

$$\lim_{\beta\to-\infty} \int \psi\, d\mu_\beta = \inf_{\mu \in \mathcal{M}(T)} \int \psi\, d\mu. \tag{5}$$

Proof of (4): We use the notation

$$p(\beta) \sim q(\beta) \quad (\beta \to \infty) \quad \text{means} \quad \lim_{\beta\to\infty} \frac{p(\beta)}{q(\beta)} = 1.$$

We have that

$$\int \psi\, d\mu_\beta = \frac{d}{d\beta} P(\varphi + \beta\psi) = \frac{d}{d\beta} \sup_{\mu \in \mathcal{M}(T)} \left\{h_\mu(T) + \int \varphi\, d\mu + \beta\int \psi\, d\mu\right\}. \tag{6}$$

Since $\mu \mapsto h_\mu(T) + \int \varphi\, d\mu$ is bounded, it is easy to see that

$$\sup_{\mu \in \mathcal{M}(T)} \left\{h_\mu(T) + \int \varphi\, d\mu + \beta\int \psi\, d\mu\right\} \sim \beta \sup_{\mu \in \mathcal{M}(T)} \int \psi\, d\mu, \tag{7}$$

so, using L'Hospital's rule applied to (6) and (7), we obtain (4). The proof of (5) is similar.

Now we prove equality with $P(\varphi, K_\alpha)$. As before, let μ_β be the Gibbs state for the potential $\varphi + \beta\psi$ where β is such that $\int \psi \, d\mu_\beta = \alpha$. Since μ_β is ergodic, by Birkhoff's ergodic theorem we get that $\mu_\beta(K_\alpha) = 1$ and this implies that (see [PP])

$$P(\varphi, K_\alpha) \geq h_{\mu_\beta}(f) + \int \varphi \, d\mu_\beta.$$

So we are left to prove

$$P(\varphi, K_\alpha) \leq P(\varphi + \beta\psi) - \beta\alpha.$$

Given $x \in X$ we use the notation $x = (x_1, x_2, ..., x_n, ...)$ and denote by $[x_1, ..., x_n]$ the cylinder set $\{y \in X : y_i = x_i, \, i = 1, ..., n\}$. Fix $\varepsilon > 0$ and, for each $N \in \mathbb{N}$, define $n_N(x)$, for $x \in K_\alpha$, to be the least integer $\geq N$ for which

$$\left| \frac{1}{n_N(x)} \sum_{i=0}^{n_N(x)-1} \psi(f^i(x)) - \alpha \right| < \varepsilon \tag{8}$$

is satisfied. Then let $\tilde{\mathcal{C}}_N^{(\varepsilon)}$ consist of all cylinders $[x_1, ..., x_{n_N(x)}]$ for $x \in K_\alpha$. Notice that $\tilde{\mathcal{C}}_N^{(\varepsilon)}$ is a countable cover of K_α, because $n_N(x) < \infty$ for each $x \in K_\alpha$ and because there are only a countable number of cylinders (of finite length) to begin with. Now take $\mathcal{C}_N^{(\varepsilon)}$ to be a subcover of $\tilde{\mathcal{C}}_N^{(\varepsilon)}$ consisting of pairwise disjoint cylinders, which exists simply because cylinders are nested. For each $C \in \mathcal{C}_N^{(\varepsilon)}$ one has that $C = [x_1^C, ..., x_{n_N(x^C)}^C]$ for some $x^C \in K_\alpha$; fixing such an x^C for each C, we define, for $\lambda \in \mathbb{R}$,

$$m(\varphi, \lambda, \mathcal{C}_N^{(\varepsilon)}) = \sum_{C \in \mathcal{C}_N^{(\varepsilon)}} \exp(-\lambda n_N(x^C) + \overline{S}_{n_N(x^C)}\varphi(x^C)),$$

where

$$\overline{S}_n\varphi(x) = \sup \left\{ \sum_{i=0}^{n-1} \varphi(f^i(y)) : y_i = x_i, \, i = 1, ..., n \right\}.$$

Then, it follows from the definition of topological pressure (see [PP]) that

$$\sup_{N \in \mathbb{N}} m(\varphi, \lambda, \mathcal{C}_N^{(\varepsilon)}) < \infty \Longrightarrow P(\varphi, K_\alpha) \leq \lambda.$$

Now, since μ_β is the Gibbs state for the potential $\varphi + \beta\psi$, there are positive constants c_1 and c_2 (independent of ε and N) such that, for every $C \in \mathcal{C}_N^{(\varepsilon)}$,

$$c_1 \leq \frac{\mu_\beta([x_1^C ... x_{n_N(x^C)}^C])}{\exp(-P(\varphi + \beta\psi) n_N(x^C) + S_{n_N(x^C)}\varphi(x^C) + \beta S_{n_N(x^C)}\psi(x^C))} \leq c_2,$$

where $S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i(x))$ (see [B2]). Summing over $C \in \mathcal{C}_N^{(\varepsilon)}$ we obtain

$$\sum_{C \in \mathcal{C}_N^{(\varepsilon)}} \exp(-P(\varphi + \beta\psi) n_N(x^C) + S_{n_N(x^C)}\varphi(x^C) + \beta S_{n_N(x^C)}\psi(x^C)) \leq c_1^{-1}. \tag{9}$$

Since φ is Hölder continuous, by a classical argument of bounded distortion, there exists a constant c such that, for every $n \in \mathbb{N}$ and $x \in X$,

$$|S_n\varphi(x) - \overline{S}_n\varphi(x)| \leq c. \tag{10}$$

So, using (8) and (10) in (9) we obtain

$$m(\varphi, P(\varphi + \beta\psi) - \beta\alpha + |\beta|\varepsilon, \mathcal{C}_N^{(\varepsilon)}) \leq e^c c_1^{-1} \quad \forall \, N \in \mathbb{N},$$

which shows that $P(\varphi, K_\alpha) \leq P(\varphi + \beta\psi) - \beta\alpha + |\beta|\varepsilon$. Since ε is arbitrarily small we get

$$P(\varphi, K_\alpha) \leq P(\varphi + \beta\psi) - \beta\alpha$$

which finishes the proof of Theorem C.

3. Main result

Let $T\colon X \to X$ and $S\colon Y \to Y$ be continous mappings of compact metric spaces, and $\pi\colon X \to Y$ be a continuous and surjective mapping such that $\pi \circ T = S \circ \pi$. Then the *relativized variational principle* (see [LW]) says that, given $\nu \in \mathcal{M}(S)$ and a continuous function $\varphi\colon X \to \mathbb{R}$,

$$\sup_{\substack{\mu \in \mathcal{M}(X) \\ \mu\circ\pi^{-1}=\nu}} \left\{ h_\mu(T) - h_\nu(S) + \int_X \varphi \, d\mu \right\} = \int_Y P(T, \varphi, \pi^{-1}(y)) \, d\nu(y), \tag{11}$$

where $P(T, \varphi, Z)$ denotes the relative pressure of T with respect to φ and a compact set $Z \subset X$. We say that μ is an *equilibrium state* for (11) if the supremum is attained at μ.

Now if, moreover, (X, T) and (Y, S) are mixing subshifts of finite type then, according to [DG], [DGH], there is a unique equilibrium state μ for (11) relative to any $\nu \in \mathcal{M}(S)$ and any Hölder continuous φ. Moreover, μ is ergodic if ν is ergodic.

Proof of Theorem A. Since φ is positive then, given $\nu \in \mathcal{M}(S)$, there is a unique real $t(\nu) \in [0, 1]$ such that

$$\int_Y P(T, -t(\nu)\varphi, \pi^{-1}(y)) \, d\nu(y) = 0 \tag{12}$$

(note that $t \mapsto P(T, -t\varphi, \pi^{-1}(y))$ is strictly decreasing). Denote by μ_ν the unique equilibrium state for (11) relative to ν and $-t(\nu)\varphi$. Then it follows from the relativized variational principle that, for $\mu \in \mathcal{M}(T)$ such that $\mu \circ \pi^{-1} = \nu$,

$$\frac{h_\mu(T) - h_\nu(S)}{\int \varphi \, d\mu} \le t(\nu) \tag{13}$$

with equality if and only if $\mu = \mu_\nu$. Put

$$D(\mu) = \frac{h_{\mu\circ\pi^{-1}}(S)}{\int \psi \circ \pi \, d\mu} + \frac{h_\mu(T) - h_{\mu\circ\pi^{-1}}(S)}{\int \varphi \, d\mu}$$

and

$$D = \sup_{\mu \in \mathcal{M}(T)} D(\mu). \tag{14}$$

Then it follows by (13) that

$$D = \sup_{\nu \in \mathcal{M}(S)} \left\{ \frac{h_\nu(S)}{\int \psi \, d\nu} + t(\nu) \right\}, \tag{15}$$

and if this supremum is attained at $\nu_0 \in \mathcal{M}_e(S)$ then the supremum in (14) is attained at $\mu_{\nu_0} \in \mathcal{M}_e(T)$ as we wish.

It follows from (15) that

$$\sup_{\nu \in \mathcal{M}(S)} \left\{ h_\nu(S) + (t(\nu) - D) \int \psi \, d\nu \right\} = 0, \tag{16}$$

and if this supremum is attained at $\nu_0 \in \mathcal{M}_e(S)$ then so is the supremum in (15) and thus the supremum in (14) is attained at $\mu_{\nu_0} \in \mathcal{M}_e(T)$ as we wish. Let

$$\underline{t} = \inf_{\nu \in \mathcal{M}(S)} t(\nu) \quad \text{and} \quad \overline{t} = \sup_{\nu \in \mathcal{M}(S)} t(\nu).$$

If $\underline{t} = \overline{t}$ then

$$D = \sup_{\nu \in \mathcal{M}(S)} \frac{h_\nu(S)}{\int \psi \, d\nu} + \overline{t} = \frac{h_{\nu_0}(S)}{\int \psi \, d\nu_0} + t(\nu_0),$$

where ν_0 is the *Gibbs state* for the potential $-D\psi$ which is well known to be ergodic (see [B3]). Otherwise, the supremum in (16) can be rewritten as

$$\sup_{\underline{t}\le t\le \overline{t}} \sup_{\substack{\nu\in\mathcal{M}(S)\\ t(\nu)=t}} \left\{ h_\nu(S) + \int (t-D)\psi\, d\nu \right\}. \tag{17}$$

According to [DG], [DGH], there is a Hölder continuous function $A_{-t\varphi}\colon Y\to\mathbb{R}$ such that

$$\int \log A_{-t\varphi}\, d\nu = \int P(T,-t\varphi,\pi^{-1}(y))\, d\nu(y), \tag{18}$$

so by (12),

$$t(\nu)=t \Leftrightarrow \int \log A_{-t\varphi}\, d\nu = 0.$$

So, the supremum in (17) can be rewritten as

$$\sup_{\underline{t}\le t\le \overline{t}} \sup \left\{ h_\nu(S) + \int (t-D)\psi\, d\nu : \nu\in\mathcal{M}(S) \text{ and } \int \log A_{-t\varphi}\, d\nu = 0 \right\}. \tag{19}$$

Now we see that we satisfy the hypotheses of Theorem B. The case for which this supremum is attained at $\underline{t}$ or $\overline{t}$ will be treated in the end of the proof, so we assume this does not occur. If $t\in(\underline{t},\overline{t})$ then there exists $\overline{\nu}\in\mathcal{M}(S)$ such that $t(\overline{\nu})<t$ which implies that

$$\begin{aligned}\inf_{\nu\in\mathcal{M}(S)} \int \log A_{-t\varphi}\, d\nu &\le \int \log A_{-t\varphi}\, d\overline{\nu} = \\ &= \int P(T,-t\varphi,\pi^{-1}(y))\, d\overline{\nu}(y) < \int P(T,-t(\overline{\nu})\varphi,\pi^{-1}(y))\, d\overline{\nu}(y) = 0.\end{aligned}$$

In the same way we get that

$$\sup_{\nu\in\mathcal{M}(S)} \int \log A_{-t\varphi}\, d\nu > 0.$$

So, applying Theorem B we get that the intermediate supremum in (19) is attained at the Gibbs measure (hence ergodic) $\nu_{\beta(t)}$ for the potential $(t-D)\psi+\beta(t)\log A_{-t\varphi}$, and the value of this supremum is, with $P(\cdot)=P(\cdot,Y)$,

$$h(t) = P((t-D)\psi + \beta(t)\log A_{-t\varphi}), \tag{20}$$

where $\beta(t)$ is the unique real satisfying

$$\int \log A_{-t\varphi}\, d\nu_{\beta(t)} = 0 \quad \text{i.e.} \quad t(\nu_{\beta(t)}) = t.$$

We shall see that the function $(\underline{t},\overline{t})\ni t\mapsto h(t)$ is continuous.

We observe the following fact from [DG] (see Corollary 4.14, Remark 4.16 and Proposition 5.5): If $\phi\colon X\to\mathbb{R}$ is Hölder continuous then there exist continuous functions $A^n_\phi\colon Y\to\mathbb{R}$, $n\in\mathbb{N}$, such that $A^n_\phi\to A_\phi$ (in the uniform topology) and $\phi\mapsto A^n_\phi$ is continuous. Moreover, if ϕ satisfies

$$|\phi(x_1)-\phi(x_2)| \le \alpha d(x_1,x_2)^\gamma, \quad x_1,x_2\in X, \tag{21}$$

then the speed of convergence of $A^n_\phi\to A_\phi$ is exponential depending only on α and γ. This implies that $[0,1]\ni t\mapsto A_{-t\varphi}$ is continuous.

Now we prove continuity of $(\underline{t},\overline{t})\ni t\mapsto \beta(t)$. Let $\mathcal{H}^{\alpha,\gamma}$ be the set of functions $\phi\colon Z\to\mathbb{R}$ satisfying (21), where Z is X or Y depending on the context.

Proposition 1. *Given $\alpha > 0$ and $\gamma \in (0,1]$ there exists $C > 0$ such that*

$$\left| \int \psi \, d\nu_{\phi_1} - \int \psi \, d\nu_{\phi_2} \right| \le C \|\phi_1 - \phi_2\|, \quad \text{for every } \psi, \phi_1, \phi_2 \in \mathcal{H}^{\alpha,\gamma},$$

where ν_ϕ is the Gibbs state for the potential ϕ.

Proof. Write $\phi_2 = \phi_1 + \delta u$ where $u = (\phi_2 - \phi_1)/\|\phi_1 - \phi_2\|$ and $\delta = \|\phi_1 - \phi_2\|$. Then, by [R1], we have that

$$\begin{aligned}\left| \int \psi \, d\nu_{\phi_1} - \int \psi \, d\nu_{\phi_2} \right| &= \left| \frac{\partial P(\phi_1 + t\psi)}{\partial t} - \frac{\partial P(\phi_1 + \delta u + t\psi)}{\partial t} \right|_{t=0} \\ &= \delta \left| \frac{\partial^2 P(\phi_1 + su + t\psi)}{\partial t \partial s} \right|_{t=0, s=\xi} = \delta Q_{\phi_1}(\psi, \xi u),\end{aligned}$$

for some $\xi \in [0, \delta]$, where Q is the bilinear form defined by (2). Now it is well known (see Chapter 5, Exercise 4 of [R1]) that the general term of the series defining Q decreases at an exponential rate $c\theta^n$ for some $c = c(\alpha, \gamma) > 0$ and $0 < \theta = \theta(\alpha, \gamma) < 1$. So take $C = c/(1-\theta)$. $\square$

The following result follows from Theorem 2.10 and Equation (5.8) of [DG] applied to the constant function 1.

Proposition 2. *For any $\gamma \in (0,1]$ there exist $\eta(\gamma) \in (0,1]$ and $C_\gamma > 0$ such that if $\phi \in \mathcal{H}^{D,\gamma}$ then $A_\phi \in \mathcal{H}^{C_\gamma D \exp(\|\phi\|), \eta(\gamma)}$.*

Let

$$F(t, \beta) = \int \log A_{-t\varphi} \, d\nu_{(t,\beta)},$$

where $\nu_{(t,\beta)}$ is the Gibbs sate for the potential $(t-D)\psi + \beta \log A_{-t\varphi}$. By Propositions 1 and 2 together with the continuity of $[0,1] \ni t \mapsto A_{-t\varphi}$, we conclude that $(t,\beta) \mapsto F(t,\beta)$ is continuous. Also, by [R1],

$$\frac{\partial F}{\partial \beta}(t, \beta) = Q_{(t-D)\psi + \beta \log A_{-t\varphi}}(\log A_{-t\varphi}, \log A_{-t\varphi}) > 0, \tag{22}$$

and, for each t, $\beta \mapsto \frac{\partial F}{\partial \beta}(t, \beta)$ is continuous. Let $t_0 \in (\underline{t}, \overline{t})$ and $\beta_0 = \beta(t_0)$. Then $F(t_0, \beta_0) = 0$ and given $\varepsilon > 0$ sufficiently small we get by (22) that

$$\text{for } t = t_0, \quad F(t, \beta_0 - \varepsilon) < 0 \quad \text{and} \quad F(t, \beta_0 + \varepsilon) > 0. \tag{23}$$

By continuity, there is $\delta > 0$ such that (23) holds for all $t \in (t_0 - \delta, t_0 + \delta)$. So, by the intermediate value theorem, there is a unique $\tilde{\beta}(t) \in (\beta_0 - \varepsilon, \beta_0 + \epsilon)$ such that $F(t, \tilde{\beta}(t)) = 0$ for all $t \in (t_0 - \delta, t_0 + \delta)$. By uniqueness we have $\beta(t) = \tilde{\beta}(t)$, which implies that $\beta(t)$ is continuous at t_0.

Finally, the continuity of $h(t)$ follows from the continuity of $t \mapsto A_{-t\varphi}$ and $\beta(t)$, together with $|P(\varphi_1) - P(\varphi_2)| \le \|\varphi_1 - \varphi_2\|$ (see [R1]). So if the supremum of $(\underline{t}, \overline{t}) \ni t \mapsto h(t)$ is attained at $t^* \in (\underline{t}, \overline{t})$ then

$$D = D(\mu_{\nu_{\beta(t^*)}}).$$

To finish, we must consider the cases for which the supremum in (17) is attained at $\underline{t}$ or $\overline{t}$. If the supremum is attained at $\underline{t}$ then

$$D \le \sup_{\nu \in \mathcal{M}(S)} \frac{h_\nu(S)}{\int \psi \, d\nu} + \underline{t} \le \frac{h_{\nu_0}(S)}{\int \psi \, d\nu_0} + t(\nu_0) \le D,$$

where ν_0 is the Gibbs measure for the potential $-D\psi$, so equality holds. Now assume that the supremum is attained at $\overline{t}$. Then by (16) and standard upper-semicontinuous arguments we get a measure $\nu \in \mathcal{M}(S)$ such that

$$h_\nu(S) + (\overline{t} - D) \int \psi \, d\nu = 0 \quad \text{and} \quad \int \log A_{-\overline{t}\varphi} \, d\nu = 0. \tag{24}$$

Let $\nu = \int \nu_\alpha \, d\alpha$ be the ergodic decomposition of ν. Since $\int \log A_{-\bar{t}\varphi} \, d\nu_\alpha \leq 0$ (because $t(\nu_\alpha) \leq \bar{t}$) and $\int \int \log A_{-\bar{t}\varphi} \, d\nu_\alpha \, d\alpha = 0$, we get that $\int \log A_{-\bar{t}\varphi} \, d\nu_\alpha = 0$ i.e $t(\nu_\alpha) = \bar{t}$ for a.e α. Then using the formula for the ergodic decomposition for the entropy in (24) we get

$$\int \left(h_{\nu_\alpha}(S) + (t(\nu_\alpha) - D) \int \psi \, d\nu_\alpha \right) d\alpha = 0.$$

Since $h_{\nu_\alpha}(S) + (t(\nu_\alpha) - D) \int \psi \, d\nu_\alpha \leq 0$ for every α, we get that

$$h_{\nu_\alpha}(S) + (t(\nu_\alpha) - D) \int \psi \, d\nu_\alpha = 0$$

for a.e. α, and thus $D = D(\mu_{\nu_\alpha})$.

□

Remark 2. By the proof of Theorem A, if the supremum (17) is not attained at the extremal points $\underline{t}$ and $\bar{t}$, then the maximizing measures of (1) are given by $\mu_{\nu_{\beta(t)}}$ where t is a zero (i.e. a maximizing point) of the function h defined by (20). So if h is C^2 and $h'' < 0$ then (1) has a *unique* maximizing measure, which is ergodic.

Note that, by (11), (18) and the classical variational principle,

$$P(\log A_\varphi + \psi, Y) = P(\varphi + \psi \circ \pi, X).$$

Is there some kind of relation for $\beta \log A_\varphi + \psi, \beta \in \mathbb{R}$?

Remark 3. In fact, (T, X) need not be a mixing subshift of finite type; what we really need is $\mathcal{F} = (X, T, Y, S, \pi)$ to be a *fibred system* which is *fibre expanding* and *topologically exact along fibres* as defined in [DG] or [DGH].

4. Measure of full dimension

Let $f \colon \mathbb{T}^2 \to \mathbb{T}^2$, $f(x, y) = (a(x, y), b(y))$ and Λ such that $f(\Lambda) = \Lambda$ be as in Theorem B. Let $\pi \colon \mathbb{T}^2 \to \mathbb{T}^1$ be the projection given by $\pi(x, y) = y$. Then $\pi \circ f = b \circ \pi$, and we are in the conditions of Theorem A with $T \equiv f|\Lambda$, $S \equiv b|\pi(\Lambda)$, $\varphi \equiv \log \partial_x a$ and $\psi \equiv \log b'$. (We have used "$\equiv$" instead of "$=$" because to be more precise one should use conjugacies to identify these maps; these conjugacies are continuous, surjective and bounded-to-one, and only fail to be a homeomorphism when some elements of the Markov partition intersect, which causes no problem when dealing with Hausdorff dimension.)

Then, using the notation of the previous section, the following is proved in [L1].

Theorem 2.

$$\dim_H \Lambda = \sup_{\nu \in \mathcal{M}_e(b|\pi(\Lambda))} \dim_H \mu_\nu$$

and if $\nu \in \mathcal{M}_e(b|\pi(\Lambda))$ *then*

$$\dim_H \mu_\nu = \frac{h_\nu(b)}{\int \log b' \, d\nu} + t(\nu).$$

Remark 4. Even though the map

$$\mathcal{M}_e(b|\pi(\Lambda)) \ni \nu \mapsto \dim_H \mu_\nu$$

is upper-semicontinuous (see Remark 6 of [L1]), we cannot conclude there is an invariant measure of full dimension because the subset $\mathcal{M}_e(b|\pi(\Lambda)) \subset \mathcal{M}(b|\pi(\Lambda))$ is not closed.

Proof of Theorem B. By the proof of Theorem A,

$$\sup_{\nu\in\mathcal{M}_e(b|\pi(\Lambda))}\left\{\frac{h_\nu(b)}{\int\log b'\,d\nu}+t(\nu)\right\}$$

is attained at some $\nu_0\in\mathcal{M}_e(b|\pi(\Lambda))$. Then, by Theorem 2,

$$\dim_{\mathrm{H}}\Lambda=\dim_{\mathrm{H}}\mu_{\nu_0}.$$

□

Remark 5. It follows from [LY] that, given $\mu\in\mathcal{M}_e(f|\Lambda)$ with $\nu=\mu\circ\pi^{-1}$,

$$\dim_{\mathrm{H}}\mu=\frac{h_\nu(b)}{\int\log b'\,d\nu}+\frac{h_\mu(f)-h_\nu(b)}{\int\log\partial_x a\,d\mu}.$$

Now by (13),

$$\frac{h_\mu(f)-h_\nu(b)}{\int\log\partial_x a\,d\mu}\le t(\nu),$$

so

$$\dim_{\mathrm{H}}\mu\le\frac{h_\nu(b)}{\int\log b'\,d\nu}+t(\nu)=\dim_{\mathrm{H}}\mu_\nu$$

with equality iff $\mu=\mu_\nu$. This shows that ergodic invariant measures of full dimension are obtain by our method.

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